TABLE 8.6 Shear, moment, slope, and deflection formulas for semi-infinite beams on elastic foundations

NOTATION: All notation is the same as that for Table 8.5. No length is defined since these beams are assumed to extend from the left end, for which restraints are defined, to a length beyond that portion affected by the loading. Note that M_A and R_A are reactions, not applied loads.

The following constants and functions, involving both beam constants and foundation constants, are hereby defined in order to permit condensing the tabulated formulas which follow

k_o = foundation modulus (unit stress per unit deflection); b_o = beam width; and $\beta = (b_o k_o / 4EI)^{1/4}$. (*Note*: See page 131 for a definition of $\langle x-a\rangle^n$.)

$F_1 = \cosh \beta x \cos \beta x$ $\qquad A_1 = 0.5e^{-\beta a} \cos \beta a$ $\qquad B_1 = 0.5e^{-\beta b} \cos \beta b$

$F_2 = \cosh \beta x \sin \beta x + \sinh \beta x \cos \beta x$ $\qquad A_2 = 0.5e^{-\beta a}(\sin \beta a - \cos \beta a)$ $\qquad B_2 = 0.5e^{-\beta b}(\sin \beta b - \cos \beta b)$

$F_3 = \sinh \beta x \sin \beta x$ $\qquad A_3 = -0.5e^{-\beta a} \sin \beta a$ $\qquad B_3 = -0.5e^{-\beta b} \sin \beta b$

$F_4 = \cosh \beta x \sin \beta x - \sinh \beta x \cos \beta x$ $\qquad A_4 = 0.5e^{-\beta a}(\sin \beta a + \cos \beta a)$ $\qquad B_4 = 0.5e^{-\beta b}(\sin \beta b + \cos \beta b)$

$F_{a1} = \langle x-a\rangle^0 \cosh \beta\langle x-a\rangle \cos \beta\langle x-a\rangle$

$F_{a2} = \cosh \beta\langle x-a\rangle \sin \beta\langle x-a\rangle + \sinh \beta\langle x-a\rangle \cos \beta\langle x-a\rangle$

$F_{a3} = \sinh \beta\langle x-a\rangle \sin \beta\langle x-a\rangle$

$F_{a4} = \cosh \beta\langle x-a\rangle \sin \beta\langle x-a\rangle - \sinh \beta\langle x-a\rangle \cos \beta\langle x-a\rangle$

$F_{a5} = \langle x-a\rangle^0 - F_{a1}$

$F_{a6} = 2\beta(x-a)\langle x-a\rangle^0 - F_{a2}$

$F_{b1} = \langle x-b\rangle^0 \cosh \beta\langle x-b\rangle \cos \beta\langle x-b\rangle$

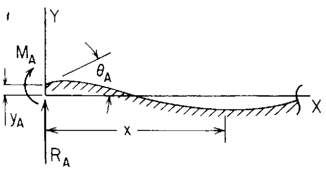
$F_{b2} = \cosh \beta\langle x-b\rangle \sin \beta\langle x-b\rangle + \sinh \langle \beta x-b\rangle \cos \beta\langle x-b\rangle$

$F_{b3} = \sinh \beta\langle x-b\rangle \sin \beta\langle x-b\rangle$

$F_{b4} = \cosh \beta\langle x-b\rangle \sin \beta - b\rangle - \sinh \beta\langle x-b\rangle \cos \beta\langle x-b\rangle$

$F_{b5} = \langle x-b\rangle^0 - F_{b1}$

$F_{b6} = 2\beta(x-b)\langle x-b\rangle^0 - F_{b2}$



Transverse shear $= V = R_A F_1 - y_A 2EI\beta^3 F_2 - \theta_A 2EI\beta^2 F_3 - M_A \beta F_4 + LT_V$

Bending moment $= M = M_A F_1 + \dfrac{R_A}{2\beta} F_2 - y_A 2EI\beta^2 F_3 - \theta_A EI\beta F_4 + LT_M$

Slope $= \theta = \theta_A F_1 + \dfrac{M_A}{2EI\beta} F_2 + \dfrac{R_A}{2EI\beta^2} F_3 - y_A \beta F_4 + LT_\theta$

Deflection $= y = y_A F_1 + \dfrac{\theta_A}{2\beta} F_2 + \dfrac{M_A}{2EI\beta^2} F_3 + \dfrac{R_A}{4EI\beta^3} F_4 + LT_y$

Expressions for R_A, M_A, θ_A, and y_A are found below for several combinations of loading and left end restraints. The loading terms LT_V, LT_M, LT_θ, and LT_y are given for each loading condition.

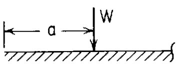
Loading, reference no. \ Left end restraint	Free	Guided	Simply supported	Fixed	Loading terms
1. Concentrated intermediate load (if $\beta a > 3$, see case 10) 	$R_A = 0 \quad M_A = 0$ $\theta_A = \dfrac{-W}{EI\beta^2} A_2$ $y_A = \dfrac{-W}{EI\beta^3} A_1$ (if $a = 0$, see case 8)	$R_A = 0 \quad \theta_A = 0$ $M_A = \dfrac{-W}{\beta} A_2$ $y_A = \dfrac{-W}{2EI\beta^3} A_4$	$M_A = 0 \quad y_A = 0$ $R_A = 2WA_1$ $\theta_A = \dfrac{W}{EI\beta^2} A_3$	$\theta_A = 0 \quad y_A = 0$ $R_A = 2WA_4$ $M_A = \dfrac{2W}{\beta} A_3$	$LT_V = -WF_{a1}$ $LT_M = \dfrac{-W}{2\beta} F_{a2}$ $LT_\theta = \dfrac{-W}{2EI\beta^2} F_{a3}$ $LT_y = \dfrac{-W}{4EI\beta^3} F_{a4}$

TABLE 8.6 Shear, moment, slope, and deflection formulas for semi-infinite beams on elastic foundations (*Continued*)

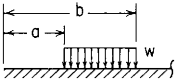
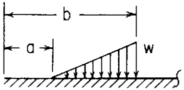
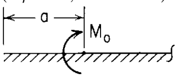
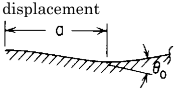
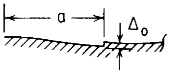
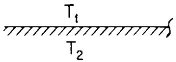
Loading, reference no. \ Left end restraint	Free	Guided	Simply supported	Fixed	Loading terms
2. Uniformly distributed load from a to b  	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{-w}{EI\beta^3}(B_3 - A_3)$ $y_A = \frac{-w}{2EI\beta^4}(B_2 - A_2)$	$R_A = 0 \quad \theta_A = 0$ $M_A = \frac{-w}{\beta^2}(B_3 - A_3)$ $y_A = \frac{w}{2EI\beta^4}(B_1 - A_1)$	$M_A = 0 \quad y_A = 0$ $R_A = \frac{w}{\beta}(B_2 - A_2)$ $\theta_A = \frac{w}{2EI\beta^3}(B_4 - A_4)$	$\theta_A = 0 \quad y_A = 0$ $R_A = \frac{-2w}{\beta}(B_1 - A_1)$ $M_A = \frac{w}{\beta^2}(B_4 - A_4)$	$LT_V = \frac{-w}{2\beta}(F_{a2} - F_{b2})$ $LT_M = \frac{-w}{2\beta^2}(F_{a3} - F_{b3})$ $LT_\theta = \frac{-w}{4EI\beta^3}(F_{a4} - F_{b4})$ $LT_y = \frac{-w}{4EI\beta^4}(F_{a5} - F_{b5})$
3. Uniform increasing load from a to b  	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{w}{2EI\beta^4}\left(\frac{B_4 - A_4}{b-a} - 2\beta B_3\right)$ $y_A = \frac{w}{2EI\beta^5}\left(\frac{B_3 - A_3}{b-a} - \beta B_2\right)$	$R_A = 0 \quad \theta_A = 0$ $M_A = \frac{w}{2\beta^3}\left(\frac{B_4 - A_4}{b-a} - 2\beta B_3\right)$ $y_A = \frac{-w}{4EI\beta^5}\left(\frac{B_2 - A_2}{b-a} - 2\beta B_1\right)$	$M_A = 0 \quad y_A = 0$ $R_A = \frac{-w}{\beta^2}\left(\frac{B_3 - A_3}{b-a} - \beta B_2\right)$ $\theta_A = \frac{w}{2EI\beta^4}\left(\frac{B_1 - A_1}{b-a} + \beta B_4\right)$	$\theta_A = 0 \quad y_A = 0$ $R_A = \frac{w}{\beta^2}\left(\frac{B_2 - A_2}{b-a} - 2\beta B_1\right)$ $M_A = \frac{w}{\beta^3}\left(\frac{B_1 - A_1}{b-a} + \beta B_4\right)$	$LT_V = \frac{-w}{2\beta^2}\left(\frac{F_{a3} - F_{b3}}{b-a} - \beta F_{b2}\right)$ $LT_M = \frac{-w}{4\beta^3}\left(\frac{F_{a4} - F_{b4}}{b-a} - 2\beta F_{b3}\right)$ $LT_\theta = \frac{-w}{4EI\beta^4}\left(\frac{F_{a5} - F_{b5}}{b-a} - \beta F_{b4}\right)$ $LT_y = \frac{-w}{8EI\beta^5}\left(\frac{F_{a6} - F_{b6}}{b-a} - 2\beta F_{b5}\right)$
4. Concentrated intermediate moment (if $\beta a > 3$, see case 11)  	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{-2M_o}{EI\beta}A_1$ $y_A = \frac{M_o}{EI\beta^2}A_4$ (if $a = 0$, see case 9)	$R_A = 0 \quad \theta_A = 0$ $M_A = -2M_oA_1$ $y_A = \frac{-M_o}{EI\beta^2}A_3$	$M_A = 0 \quad y_A = 0$ $R_A = -2M_o\beta A_4$ $\theta_A = \frac{M_o}{EI\beta}A_2$	$\theta_A = 0 \quad y_A = 0$ $R_A = 4M_o\beta A_3$ $M_A = 2M_oA_2$	$LT_V = -M_o\beta F_{a4}$ $LT_M = M_oF_{a1}$ $LT_\theta = \frac{M_o}{2EI\beta}F_{a2}$ $LT_y = \frac{M_o}{2EI\beta^2}F_{a3}$
5. Externally created concentrated angular displacement  	$R_A = 0 \quad M_A = 0$ $\theta_A = -2\theta_oA_4$ $y_A = \frac{-2\theta_o}{\beta}A_3$	$R_A = 0 \quad \theta_A = 0$ $M_A = -2\theta_oEI\beta A_4$ $y_A = \frac{\theta_o}{\beta}A_2$	$M_A = 0 \quad y_A = 0$ $R_A = 4\theta_oEI\beta^2A_3$ $\theta_A = -2\theta_oA_1$	$\theta_A = 0 \quad y_A = 0$ $R_A = -4\theta_oEI\beta^2A_2$ $M_A = -4\theta_oEI\beta A_1$	$LT_V = -2\theta_oEI\beta^2F_{a3}$ $LT_M = -\theta_oEI\beta F_{a4}$ $LT_\theta = \theta_oF_{a1}$ $LT_y = \frac{\theta_o}{2\beta}F_{a2}$

TABLE 8.6 Shear, moment, slope, and deflection formulas for semi-infinite beams on elastic foundations (*Continued*)

6. Externally created concentrated lateral displacement 	$R_A = 0 \quad M_A = 0$ $\theta_A = 4\Delta_o \beta A_3$ $y_A = 2\Delta_o A_2$	$R_A = 0 \quad \theta_A = 0$ $M_A = 4\Delta_o EI\beta^2 A_3$ $y_A = -2\Delta_o A_1$	$M_A = 0 \quad y_A = 0$ $R_A = -4\Delta_o EI\beta^3 A_2$ $\theta_A = -2\Delta_o \beta A_4$	$\theta_A = 0 \quad y_A = 0$ $R_A = 8\Delta_o EI\beta^3 A_1$ $M_A = -4\Delta_o EI\beta^2 A_4$	$LT_V = -2\Delta_o EI\beta^3 F_{a2}$ $LT_M = -2\Delta_o EI\beta^2 F_{a3}$ $LT_\theta = -\Delta_o \beta F_{a4}$ $LT_y = \Delta_o F_{a1}$
7. Uniform temperature differential from top to bottom 	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{T_1 - T_2}{t\beta}\gamma$ $y_A = -\frac{T_1 - T_2}{2t\beta^2}\gamma$	$R_A = 0 \quad \theta_A = 0$ $M_A = \frac{T_1 - T_2}{t}\gamma EI$ $y_A = 0$	$M_A = 0 \quad y_A = 0$ $R_A = \frac{T_1 - T_2}{t}\gamma EI\beta$ $\theta_A = \frac{T_1 - T_2}{2t\beta}\gamma$	$\theta_A = 0 \quad y_A = 0$ $R_A = 0$ $M_A = \frac{T_1 - T_2}{t}\gamma EI$	$LT_V = \frac{T_1 - T_2}{t}\gamma EI\beta F_4$ $LT_M = \frac{T_1 - T_2}{t}\gamma EI(1 - F_1)$ $LT_\theta = -\frac{T_1 - T_2}{2t\beta}\gamma F_2$ $LT_y = -\frac{T_1 - T_2}{2t\beta^2}\gamma F_3$

Simple loads on semi-infinite and on infinite beams on elastic foundations

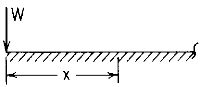
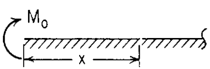
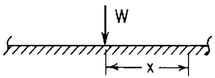
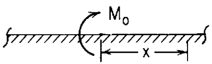
Loading, reference no.	Shear, moment, and deformation equations	Selected maximum values
8. Concentrated end load on a semi-infinite beam, left end free 	$V = -We^{-\beta x}(\cos\beta x - \sin\beta x)$ $M = -\frac{W}{\beta}e^{-\beta x}\sin\beta x$ $\theta = \frac{W}{2EI\beta^2}e^{-\beta x}(\cos\beta x + \sin\beta x)$ $y = -\frac{W}{2EI\beta^3}e^{-\beta x}\cos\beta x$	Max $V = -W$ at $x = 0$ Max $M = -0.3224\frac{W}{\beta}$ at $x = \frac{\pi}{4\beta}$ Max $\theta = \frac{W}{2EI\beta^2}$ at $x = 0$ Max $y = \frac{-W}{2EI\beta^3}$ at $x = 0$
9. Concentrated end moment on a semi-infinite beam, left end free 	$V = -2M_o\beta e^{-\beta x}\sin\beta x$ $M = M_o e^{-\beta x}(\cos\beta x + \sin\beta x)$ $\theta = -\frac{M_o}{EI\beta}e^{-\beta x}\cos\beta x$ $y = -\frac{M_o}{2EI\beta^2}e^{-\beta x}(\sin\beta x - \cos\beta x)$	Max $V = -0.6448M_o\beta$ at $x = \frac{\pi}{4\beta}$ Max $M = M_o$ at $x = 0$ Max $\theta = -\frac{M_o}{EI\beta}$ at $x = 0$ Max $y = \frac{M_o}{2EI\beta^2}$ at $x = 0$

TABLE 8.6 Shear, moment, slope, and deflection formulas for semi-infinite beams on elastic foundations (*Continued*)

Loading, reference no.	Shear, moment, and deformation equations	Selected maximum values
10. Concentrated load on an infinite beam  	$V = -\frac{W}{2} e^{-\beta x} \cos \beta x$ $M = \frac{W}{4\beta} e^{-\beta x}(\cos \beta x - \sin \beta x)$ $\theta = \frac{W}{4EI\beta^2} e^{-\beta x} \sin \beta x$ $y = -\frac{W}{8EI\beta^3} e^{-\beta x}(\cos \beta x + \sin \beta x)$	$\text{Max } V = -\frac{W}{2}$ at $x = 0$ $\text{Max } M = \frac{W}{4\beta}$ at $x = 0$ $\text{Max } \theta = 0.0806 \frac{W}{EI\beta^2}$ at $x = \frac{\pi}{4\beta}$ $\text{Max } y = -\frac{W}{8EI\beta^3}$ at $x = 0$
11. Concentrated moment on an infinite beam  	$V = -\frac{M_o\beta}{2} e^{-\beta x}(\cos \beta x + \sin \beta x)$ $M = \frac{M_o}{2} e^{-\beta x} \cos \beta x$ $\theta = -\frac{M_o}{4EI\beta} e^{-\beta x}(\cos \beta x - \sin \beta x)$ $y = -\frac{M_o}{4EI\beta^2} e^{-\beta x} \sin \beta x$	$\text{Max } V = -\frac{M_o\beta}{2}$ at $x = 0$ $\text{Max } M = \frac{M_o}{2}$ at $x = 0$ $\text{Max } \theta = -\frac{M_o}{4EI\beta}$ at $x = 0$ $\text{Max } y = -0.0806 \frac{M_o}{EI\beta^2}$ at $x = \frac{\pi}{4\beta}$